

# Exact Supersymmetry on the Lattice

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## Abstract

*We discuss the possibility of representing supersymmetry exactly in a lattice discretized system. In particular, we construct a perfect supersymmetric action for the Wess-Zumino model.*

## 1 Introduction

Lattice simulations of supersymmetric systems usually apply formulations, which reveal the supersymmetry in the continuum limit but not in the lattice discretized version [1]. This note addresses the issue of a lattice action, which directly displays a continuous form of supersymmetry. Some related works are listed in Ref. [2].

### 1.1 A simple supersymmetric model

We first consider a simple 2d SUSY model [3] given in the continuous Euclidean space by the Lagrangian

$$\mathcal{L} = \bar{\psi} \gamma_\mu \partial_\mu \psi + \partial_\mu \varphi \partial_\mu \varphi , \quad (1.1)$$

where  $\psi$  is a real “Majorana spinor”<sup>1</sup> and  $\varphi$  is a real scalar field. Many qualitative features with respect to a lattice formulation are the same as in the Wess-Zumino model. The action  $S$  is invariant under the field transformations

$$\delta\psi = -\gamma_\mu \partial_\mu \varphi \epsilon \quad , \quad \delta\varphi = \bar{\epsilon} \psi \quad , \quad (1.2)$$

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<sup>1</sup>Strictly speaking, there are no Majorana spinors in Euclidean space, we just refer to the Euclidean version of the corresponding formulae in Minkowski space. A definition is given for instance in Ref. [4], based on a Euclidean analog of charge conjugation. Note that  $\bar{\psi}$  and  $\psi$  are not independent.

where the two components of the transformation parameter  $\epsilon$  are real Grassmann variables. As an important property, we note that the supersymmetric generator forms a closed algebra with the translation operator,

$$[\delta_1, \delta_2]\varphi = (\bar{\epsilon}_1 \gamma_\mu \epsilon_2 - \bar{\epsilon}_2 \gamma_\mu \epsilon_1) \partial_\mu \varphi . \quad (1.3)$$

## 1.2 Ansatz for a lattice formulation

Let us consider a rather general ansatz for a lattice discretization of this free system in momentum space,

$$\begin{aligned} S &= \int_B \frac{d^2 p}{(2\pi)^2} \left\{ \bar{\Psi}(-p) [\gamma_\mu \rho_\mu(p) + \lambda(p)] \Psi(p) + \Phi(-p) \Omega(p) \Phi(p) \right\} , \\ \delta \Psi(p) &= -[\gamma_\mu R_\mu(p) + L(p)] \Phi(p) \epsilon , \\ \delta \Phi(p) &= \bar{\epsilon} [u(p) + \gamma_\mu v_\mu(p)] \Psi . \end{aligned} \quad (1.4)$$

Here,  $\Psi$  and  $\Phi$  are the massless lattice fermion resp. scalar field, and the new quantities, which we introduce as an ansatz for the inverse propagators and for the transformation terms  $(\rho_\mu, \lambda, \Omega, R_\mu, L, u, v_\mu)$  are real in coordinate space. It is desirable that they are all local, i.e. analytic in momentum space. We require the low energy expansion of the action to reproduce the correct continuum limit, and the inverse propagators obey in coordinate space the normalization conditions <sup>2</sup>

$$\sum_x x_\mu \rho_\mu(x) = 1 , \quad \sum_x \lambda(x) = 0 , \quad \sum_x x^2 \Omega(x) = -4 . \quad (1.5)$$

We do not require the lattice transformation terms to correspond exactly to the continuum transformations (1.2). Hence this ansatz includes a possible “remnant supersymmetry” of the lattice action, similar to the Ginsparg-Wilson relation for the chiral symmetry [6]. The general (remnant) supersymmetry requirement  $\delta S = 0$  amounts to

$$\begin{aligned} -R_\mu(-p) [\rho_\mu(p) - \rho_\mu(-p)] + L(-p) [\lambda(-p) - \lambda(p)] + u(p) [\Omega(p) + \Omega(-p)] &= 0 , \\ R_\mu(-p) [\lambda(-p) - \lambda(p)] + L(-p) [\rho_\mu(-p) - \rho_\mu(p)] + v_\mu(p) [\Omega(p) + \Omega(-p)] &= 0 . \end{aligned} \quad (1.6)$$

It is sensible to assume the following symmetry properties in the action:  $\rho_\mu$  is odd in the  $\mu$  component and even in all other directions, while the Dirac scalars  $\lambda$  and  $\Omega$  are entirely even. Then the conditions simplify to

$$\begin{aligned} -R_\mu(-p) \rho_\mu(p) + u(p) \Omega(p) &= 0 , \\ -L(-p) \rho_\mu(p) + v_\mu(p) \Omega(p) &= 0 . \end{aligned} \quad (1.7)$$

Remarkably,  $\lambda$  does not occur any more in these conditions.

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<sup>2</sup>In the first expression, there is no sum over  $\mu$ .

Finally, we assume for the transformation terms  $R_\mu$  and  $L$  the same symmetries as for  $\rho_\mu, \lambda$ , respectively, which leads to

$$\begin{aligned} -R_\mu(p)\rho_\mu(p) &= u(p)\Omega(p) , \\ L(p)\rho_\mu(p) &= v_\mu(p)\Omega(p) \quad (\mu = 1, 2). \end{aligned} \quad (1.8)$$

The translation operator is identified from

$$\begin{aligned} [\delta_1, \delta_2]\Phi_x &= \sum_y [\bar{\epsilon}_1 Q(x-y) \epsilon_2 - \bar{\epsilon}_2 Q(x-y) \epsilon_1] \Phi_y , \\ Q(x-y) &= \sum_z [u(x-z) + \gamma_\nu v_\nu(x-z)] [\gamma_\mu R_\mu(z-y) + L(z-y)] , \end{aligned} \quad (1.9)$$

but this general form is not immediately instructive.

Let us consider simple solutions for the case  $u = 1, v_\mu = L = 0$ .<sup>3</sup> The standard lattice action,  $\rho_\mu(p) = i\bar{p}_\mu := i \sin p_\mu$ ,  $\Omega(p) = \hat{p}^2 := \sum_\mu [2 \sin(p_\mu/2)]^2$ , requires

$$R_\mu(p) = \frac{\hat{p}^2}{\bar{p}^2} i\bar{p}_\mu ,$$

which is singular at  $p = (\pi, 0)$ ,  $(0, \pi)$  and  $(\pi, \pi)$ , hence the transformation is non-local. An obvious concept to simplify  $R_\mu$  – and to obtain the same dispersion relation for fermion and scalar – is the use of the same lattice differential operator for the scalar and the fermion part of the action.

One way to do so is to set  $\Omega(p) = \bar{p}^2$ ,  $R_\mu(p) = \rho_\mu(p) = i\bar{p}_\mu$ , which is local but affected by doubling, both, for the fermion as well as the scalar. In the present model, unlike the case of the Wess-Zumino model [5], we cannot treat them by adding Wilson terms  $((r/2) \cdot \hat{p}^2)$ , because terms of this kind alter  $\Omega$  but not  $\rho_\mu$  (the fermionic Wilson terms contributes to  $\lambda$ ). Hence  $R_\mu$  gets complicated and non-local again,  $R_\mu(p) = i\bar{p}_\mu(1 + (r/2)\hat{p}^2/\bar{p}^2)$ .

If one is ready to accept non-locality, then it looks simpler to adjust the differential operators the other way round,  $\rho_\mu(p) = R_\mu(p) = i\hat{p}_\mu$ , as suggested in Ref. [7], and  $\Omega(p) = \hat{p}^2$ . Then the translation operator resulting from the lattice version of eq. (1.3) corresponds to a half-lattice shift, whereas it is a full lattice shift for the option mentioned before. However, the fermionic inverse propagator performs a finite gap at the edge of the Brillouin zone, so we are dealing with a non-locality similar to the SLAC fermions. This suggests that also this approach fails to recover Lorentz invariance in the presence of a gauge interaction, as was pointed out for the SLAC fermions on the one loop level [8].

One can construct a number of solutions of this type by hand. For instance, the standard action together with  $u(p) = \prod_\mu \cos(p_\mu/2)$  even provides locality, but such hand-made constructions look hardly satisfactory. Similar to the problem of chiral fermions on the lattice, they do not appear promising for the consistent incorporation of interactions. Hence we are going to follow a different, more systematic strategy.

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<sup>3</sup>Note that the two transformation terms with an unusual Dirac structure,  $L$  and  $v_\mu$ , can only come into play simultaneously.

## 2 A perfect supersymmetric lattice action

Since we are considering a free theory here, we can construct a perfect lattice action by “blocking from the continuum”, which corresponds to a block variable renormalization group transformation (RGT) with blocking factor infinity,

$$e^{-S[\Psi, \Phi]} = \int D\psi D\varphi e^{-s[\psi, \varphi] - T[\Psi, \psi, \Phi, \varphi]} , \quad (2.1)$$

where  $S$  is the perfect lattice action (i.e. an action without lattice artifacts),  $s$  the continuum action, and  $T$  the transformation term. We choose the latter such that the functional integral remains Gaussian,

$$\begin{aligned} T &= \sum_{x,y} [\bar{\Psi}_x - \int_{C_x} \bar{\psi}(u) du] (\alpha^f)_{xy}^{-1} [\Psi_y - \int_{C_y} \psi(u) du] \\ &+ \sum_{x,y} [\Phi_x - \int_{C_x} \varphi(u) du] (\alpha^s)_{xy}^{-1} [\Phi_y - \int_{C_y} \varphi(u) du] , \end{aligned} \quad (2.2)$$

where  $C_x$  is the unit square with center  $x$ , and  $\alpha^f, \alpha^s$  are arbitrary RGT parameters ( $\alpha^s$  has to be positive). The resulting perfect action reads

$$\begin{aligned} S[\Psi, \Phi] &= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} d^2p \left\{ \bar{\Psi}(-p) \Delta^f(p)^{-1} \Psi(p) + \Phi(-p) \Delta^s(p)^{-1} \Phi(p) \right\} \\ \Delta^f(p) &= \sum_{l \in \mathbf{Z}^2} \frac{\Pi(p + 2\pi l)^2}{i(p_\mu + 2\pi l_\mu) \gamma_\mu} + \alpha^f(p) , \quad \Delta^s(p) = \sum_{l \in \mathbf{Z}^2} \frac{\Pi(p + 2\pi l)^2}{(p + 2\pi l)^2} + \alpha^s(p) , \end{aligned} \quad (2.3)$$

where  $\Pi(p) := \Pi_\mu \hat{p}_\mu / p_\mu$ . Locality requires  $\alpha^f \neq 0$ , which naively breaks the chiral symmetry. However, the latter is still present in the observables [9], and a continuous remnant form of it even persists in the lattice action, if  $\alpha^f(p)$  is analytic [6].<sup>4</sup>

Now we consider the SUSY transformation. The variation of the continuum fields is given in eq. (1.2). If we transform simultaneously the lattice fields as

$$\delta \Psi_x = -\gamma_\mu \int_{C_x} \partial_\mu \varphi(u) du \epsilon , \quad \delta \Phi_x = \bar{\epsilon} \int_{C_x} \psi(u) du , \quad (2.4)$$

then all the square brackets in the expression for  $T$  (eq. (2.2)) remain invariant – and so does the continuum action – hence  $\delta S = 0$ .

Everything is consistent since we block the fields as well as their variations from the continuum. Note that this is not a solution along the lines of section 1.2, because the transformations of the lattice fields are not expressed directly in terms of lattice fields. (The special case of a  $\delta$  function RGT,  $\alpha^s, \alpha^f \rightarrow 0$ , is an exception, see below.) Hence the solution is somehow implicit.

However, we can re-write the field variations in terms of lattice variables. First, we define a continuum current

$$j_\mu = \gamma_\mu \varphi . \quad (2.5)$$

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<sup>4</sup>In this case, the full chiral symmetry of the fermion propagator is broken only locally. This is the property denoted as Ginsparg-Wilson relation.

We now block this current by integrating the flux through the face between two adjacent lattice cells,

$$J_{\mu,x} = \int_{-1/2}^{1/2} j_{\mu}(x + \frac{1}{2}\hat{\mu} + u_{\nu}) du_{\nu} \quad (\nu \neq \mu, |\hat{\mu}| = 1). \quad (2.6)$$

This is a perfect lattice current [9]. Here we assume it to be implemented so that eq. (2.6) holds exactly. As an interesting property, its lattice divergence is equal to the blocked continuum divergence,

$$\nabla_{\mu} J_{x,\mu} := \sum_{\mu} (J_{\mu,x} - J_{\mu,x-\hat{\mu}}) = \int_{C_x} \partial_{\mu} j_{\mu}(u) du \quad (\text{Gauss' law}). \quad (2.7)$$

We are now prepared to write the variation of the lattice fermion field from eq. (2.4) in terms of lattice variables,

$$\delta\Psi = -\nabla_{\mu} J_{\mu} \epsilon, \quad \delta\bar{\Psi} = -\bar{\epsilon} \nabla_{\mu} J_{\mu}. \quad (2.8)$$

In addition, we introduce the fermionic lattice field

$$\tilde{\Psi}_x := \int_{C_x} \psi(u) du, \quad (2.9)$$

which allows us to write also  $\delta\Phi$  in terms of lattice quantities,

$$\delta\Phi = \bar{\epsilon} \tilde{\Psi}. \quad (2.10)$$

For a  $\delta$  function RGT in the fermionic sector we have  $\Psi = \tilde{\Psi}$ , but for finite  $\alpha^f$  this does not hold exactly.

A generalization is possible for instance with respect to the blocking scheme. Instead of the block average scheme we have used so far, we can start from a more general ansatz

$$\begin{aligned} T = & \sum_{x,y} [\bar{\Psi}_x - \int \Pi^f(x-u) \bar{\psi}(u) du] (\alpha^f)_{xy}^{-1} [\Psi_y - \int \Pi^f(x-u) \psi(u) du] \\ & + \sum_{x,y} [\Phi_x - \int \Pi^s(x-u) \varphi(u) du] (\alpha^s)_{xy}^{-1} [\Phi_y - \int \Pi^s(x-u) \varphi(u) du], \end{aligned} \quad (2.11)$$

where  $\int \Pi^f(u) du = \int \Pi^s(u) du = 1$ . Both convolution functions  $\Pi^f, \Pi^s$  are peaked around zero and decay fast enough to preserve locality on the lattice.

Correspondingly, the variations of the lattice fields turn into

$$\begin{aligned} \delta\Psi_x &= -\gamma_{\mu} \int \Pi^f(x-u) \partial_{\mu} \varphi(u) du \epsilon, \\ \delta\Phi_x &= \bar{\epsilon} \int \Pi^s(x-u) \psi(u) du := \bar{\epsilon} \tilde{\Psi}_x, \end{aligned} \quad (2.12)$$

where we have adjusted the definition of  $\tilde{\Psi}$ . If we want to achieve  $\Psi = \tilde{\Psi}$ , then we need – except for  $\alpha^f = 0$  – also  $\Pi^f = \Pi^s$ .

This generalized scheme does not yield an obvious lattice current any more; the latter is a virtue of the block average scheme (characterized by  $\Pi^f(u)$ ,  $\Pi^s(u) = 1$  if  $u \in C_0$ , and 0 otherwise). In the general case, it is easier to consider the continuum divergence

$$d(u) = \partial_\mu j_\mu(u) = \gamma_\mu \partial_\mu \varphi(u) , \quad (2.13)$$

and build from it directly the lattice divergence

$$D_x = \int \Pi^f(x-u) d(u) du , \quad \delta \Psi = -D \epsilon . \quad (2.14)$$

Regarding the transformation algebra, we obtain

$$[\delta_1, \delta_2] \Phi_x = (\bar{\epsilon}_1 \gamma_\mu \epsilon_2 - \bar{\epsilon}_2 \gamma_\mu \epsilon_1) \int \Pi^s(x-u) \partial_\mu \varphi(u) du . \quad (2.15)$$

In particular, for the block average scheme this simplifies to

$$[\delta_1, \delta_2] \Phi = \bar{\epsilon}_1 \nabla_\mu J_\mu \epsilon_2 - \bar{\epsilon}_2 \nabla_\mu J_\mu \epsilon_1 . \quad (2.16)$$

We see that the continuum translation operator is inherited by the perfect lattice formulation in a consistent way: eqs. (2.15), (2.16) show that the corresponding lattice translation operator is just the blocked continuum translation operator. The formula for  $[\delta_1, \delta_2] \Psi$  is analogous. If we require the resulting translation operators to be identical, then we need  $\Pi^f = \Pi^s$ . Then the algebra of field variations and translation closes under the blocking integral.

In any case, the fermionic and scalar spectrum are equal, because they both coincide with the continuum spectrum.

It is straightforward to apply this perfect lattice formulation to more complicated cases, see below. Interactions can be included perturbatively in the process of blocking from the continuum. For asymptotically free theories, at  $m = 0$  even the classically perfect action behaves perfectly [10]. Hence by means of an implicit (but not just symbolic) definition of the action – in terms of classical inverse blocking – we can also go beyond perturbation theory in the massless case. This is analogous to the fixed point formulation of a chiral gauge theory on the lattice [9, 11].

### 3 Adding an auxiliary scalar field

To proceed to the 2d Wess-Zumino model, we include an auxiliary scalar field.<sup>5</sup> This equilibrates the number of fermionic and bosonic degrees of freedom. The continuum Lagrangian and the field variations read

$$\begin{aligned} \mathcal{L} &= \bar{\psi} \gamma_\mu \partial_\mu \psi + \partial_\mu \varphi \partial_\mu \varphi + f^2 , \\ \delta \psi &= -(\gamma_\mu \partial_\mu \varphi + f) \epsilon , \quad \delta \varphi = \bar{\epsilon} \psi , \quad \delta f = \bar{\epsilon} \gamma_\mu \partial_\mu \psi . \end{aligned} \quad (3.1)$$

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<sup>5</sup>This is even necessary for the general validity of the resulting translation operator; otherwise it only holds on-shell. See for instance P. Freud, “Introduction to Supersymmetry”, Cambridge University Press, 1986.

Now the method presented in Ref. [5] is applicable in an extended form, if we use the following lattice discretization:

$$\begin{aligned}
S &= \int_B \frac{d^2 p}{(2\pi)^2} \left[ \bar{\Psi}(-p) i\gamma_\mu \bar{p}_\mu \Psi(p) + \Phi(-p) \bar{p}^2 \Phi(p) + F(-p) F(p) \right. \\
&\quad \left. + \bar{\Psi}(-p) W^f(p) \Psi(p) + 2\Phi(-p) W^s(p) F(p) - \Phi(-p) W^s(p)^2 \Phi(p) \right] , \\
\delta\Psi(p) &= -\{[i\gamma_\mu \bar{p}_\mu + W^s(p)]\Phi(p) + F(p)\} \epsilon \\
\delta\Phi(p) &= \bar{\epsilon} \Psi(p) \\
\delta F(p) &= \bar{\epsilon} [i\gamma_\mu \bar{p}_\mu - W^s(p)] \Psi(p) ,
\end{aligned} \tag{3.2}$$

where  $W^f(p)$ ,  $W^s(p)$  are some sort of Wilson terms (zero at the origin, non-zero at the edges of the Brillouin zone, local and  $2\pi$  periodic, which implies that they are even). Hence they remove the degeneracy of the physical particles with their doublers. The standard form  $1/2 \hat{p}^2$  is an example, but we can also insert more general scalar and fermionic Wilson terms<sup>6</sup> and we always arrive at  $\delta S = 0$ . If we also want the fermion and scalar spectrum to coincide, then we have to relate  $W^f$  and  $W^s$ . The procedure applied in Ref. [5] further restores a remnant chiral symmetry by means of the so-called overlap formalism, and this could also be done here.

Instead we can apply the perfect action machine from section 2, starting from the continuum system (3.1). This also solves the doubling problem and maintains a (remnant) chiral symmetry. The perfect propagator of the lattice field  $F$  reads

$$\Delta^{\bar{s}}(p) = \sum_{l \in \mathbf{Z}^2} \Pi^{\bar{s}}(p + 2\pi l)^2 + \alpha^{\bar{s}}(p) , \tag{3.3}$$

where  $\alpha^{\bar{s}}$  is a RGT parameter analogous to  $\alpha^s$ . We now introduce *two* continuum currents,  $\gamma_\mu \varphi$  and  $\gamma_\mu \psi$ , and we construct perfect currents from them. The explicit formulae are a straightforward extension of the formulae in section 2. We do not display them here, but we write them down for the further extension to the 4d Wess-Zumino model in the appendix.

## 4 Conclusions

We illuminated the problem of a direct construction of a supersymmetric lattice formulation. Then we have shown how a construction in terms of renormalization group transformations can be achieved. We preserve invariance under a continuous supersymmetric type of field transformations in a local perfect lattice action, which has also a remnant chiral symmetry. This applies to the 2d models discussed above, as well as to the 4d Wess-Zumino model, see appendix. We remark that the perfect

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<sup>6</sup>The continuum limit of this action does in fact correspond to eq. (3.1), as we see if we substitute  $F(p) \rightarrow \tilde{F}(p) = F(p) + \Phi(p) W^s(p)$ . Then the mixed term of  $F$  and  $\Phi$  disappears, and we obtain another irrelevant term  $-\Phi(-p) W^s(p)^2 \Phi(p)$ .

formulation also cures the well-known problems related to the Leibniz rule [2]<sup>7</sup> – which breaks down for usual lattice discretizations – because here we keep track of the exact continuum differential operators. This is manifest in the translation operator, which results from a commutator of field variations: in the perfect lattice formulation, we obtain the consistently blocked continuum translation operator. Therefore the algebra with the field variations closes.

Moreover, in the perfect lattice formulation, the fermionic and scalar dispersion relation coincide automatically.

The next step is the inclusion of the gauge interaction; this work is in progress. A consistent blocking of the gauge fields from the continuum leads to a perfect action with all the continuum symmetry properties in the observables – and also in the action, if the transformation term respects these symmetries – but this construction can only be performed perturbatively. For asymptotically free theories in the massless case, a classically perfect action – constructed by simplified inverse blocking (based on minimization) – is sufficient for the same purpose and enables the step beyond perturbation theory.

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## A Application to the 4d Wess-Zumino model

For completeness, we write down the corresponding formulae for a perfect lattice formulation of the 4d Wess-Zumino model:

$$e^{-S[\Psi, \Phi^{(1)}, \Phi^{(2)}, F^{(1)}, F^{(2)}]} = \int D\psi D\varphi^{(1)} D\varphi^{(2)} Df^{(1)} Df^{(2)} e^{-s[\psi, \varphi^{(1)}, \varphi^{(2)}, f^{(1)}, f^{(2)}]} \times \exp\{-T[\Psi, \psi, \Phi^{(1)}, \varphi^{(1)}, \Phi^{(2)}, \varphi^{(2)}, F^{(1)}, f^{(1)}, F^{(2)}, f^{(2)}]\} , \quad (\text{A.1})$$

$$s[\psi, \varphi^{(1)}, \varphi^{(2)}, f^{(1)}, f^{(2)}] = \frac{1}{2} \int d^4x [\bar{\psi} \gamma_\mu \partial_\mu \psi + (\partial_\mu \varphi^{(1)})^2 + (\partial_\mu \varphi^{(2)})^2 + f^{(1)2} + f^{(2)2}] ,$$

$$T = \sum_{x,y} [\bar{\Psi}_x - \int_{C_x} \bar{\psi}(u) du] (\alpha^f)_{xy}^{-1} [\Psi_y - \int_{C_y} \psi(u) du] + \sum_{i=1}^2 \sum_{x,y} \left\{ [\Phi_x^{(i)} - \int_{C_x} \varphi^{(i)}(u) du] (\alpha_i^s)_{xy}^{-1} [\Phi_y^{(i)} - \int_{C_y} \varphi^{(i)}(u) du] + [F_x^{(i)} - \int_{C_x} f^{(i)}(u) du] (\alpha_i^{\bar{s}})_{xy}^{-1} [F_y^{(i)} - \int_{C_y} f^{(i)}(u) du] \right\} , \quad (\text{A.2})$$

where  $\alpha_i^s$  and  $\alpha_i^{\bar{s}}$  have to be positive, and we are using the block average scheme. The perfect action  $S$  can be assembled by analogy from eqs. (2.3) and (3.3). The transformation of the continuum fields have the usual form,

$$\begin{aligned} \delta\psi &= -[\gamma_\mu(\partial_\mu \varphi^{(1)} + \gamma_5 \partial_\mu \varphi^{(2)}) + f^{(1)} + \gamma_5 f^{(2)}] \epsilon , \\ \delta\varphi^{(1)} &= \bar{\epsilon} \psi , \quad \delta\varphi^{(2)} = \bar{\epsilon} \gamma_5 \psi , \end{aligned}$$

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<sup>7</sup>See in particular the first paper by S. Nojiri.



$$\delta f^{(1)} = \bar{\epsilon} \gamma_\mu \partial_\mu \psi, \quad \delta \varphi^{(2)} = \bar{\epsilon} \gamma_5 \gamma_\mu \partial_\mu \psi, \quad (\text{A.3})$$

and the lattice field transformations amount to

$$\begin{aligned} \delta \Psi_x &= - \int_{C_x} \left\{ \gamma_\mu [\partial_\mu \varphi^{(1)}(u) + \gamma_5 \partial_\mu \varphi^{(2)}(u)] + f^{(1)}(u) + \gamma_5 f^{(2)}(u) \right\} du \in \\ &:= - \left[ \nabla_\mu [J_{\mu,x} + J_{\mu,x}^5] + \tilde{F}_x^{(1)} + \gamma_5 \tilde{F}_x^{(2)} \right] \in, \\ \delta \Phi_x^{(1)} &= \bar{\epsilon} \int_{C_x} \psi(u) du = \bar{\epsilon} \tilde{\Psi}_x, \quad \delta \Phi_x^{(2)} = \bar{\epsilon} \gamma_5 \int_{C_x} \psi(u) du = \bar{\epsilon} \gamma_5 \tilde{\Psi}_x, \\ \delta F_x^{(1)} &= \bar{\epsilon} \gamma_\mu \int_{C_x} \partial_\mu \psi(u) du := \bar{\epsilon} \nabla_\mu I_{\mu,x}, \\ \delta F_x^{(2)} &= \bar{\epsilon} \gamma_5 \gamma_\mu \int_{C_x} \partial_\mu \psi(u) du = \bar{\epsilon} \gamma_5 \nabla_\mu I_{\mu,x}, \end{aligned} \quad (\text{A.4})$$

where  $J_\mu$ ,  $J_\mu^5$  and  $I_\mu$  are perfect lattice currents.

Again the resulting translation operator is the blocked continuum translation operators, it forms a closed algebra with the field variations, the Leibniz rule is satisfied, and the dispersion relations of  $\Psi$  and  $\Phi$  coincide.

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